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ASYMPTOTIC PROPERTIES OF VOLTERRA EQUATIONS WITH NONINTEGRABLE --ETC(U)
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ASYMPTOTIC PROPERTIES OF VOLTERRA EQUATIONS WITH NONINTEGRABLE KERNELS

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ABSTRACT

We study the asymptotic behavior of the solutions of the scalar Volterra integrodifferential equation

$$\text{(E)} \qquad x'(t) + (a * g(x))(t) = f(t), \quad t \geq 0 , \quad x(0) = x_0 ,$$

where $a,f : R^+ \to R$ and $g:R \to R$ are given functions, $*$ denotes convolution and $x : R^+ \to R$ is the solution. We are in particular interested in the largely unsolved case when $a \notin L^1(R^+)$ and f vanishes at infinity but does not belong to any $L^p(R^+)$ space for $p < \infty$. The report examines both the linear $(g(x) \equiv x)$ and the nonlinear $(g(x) \neq x)$ version of (E).

AMS(MOS) Subject Classification: 45D05, 45M05, 45G10

Key Words: Volterra equations, nonlinear integral equations, asymptotic behavior, frequency domain methods

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SIGNIFICANCE AND EXPLANATION

In the construction of mathematical models of technical and physical systems one frequently ends up with equations in which the current rate of change $\left(= \frac{dx(t)}{dt}\right)$ of the state of the system $(= x(t))$ is a function not only of $x(t)$ but also of $x(\tau)$ for past times $\tau < t$. Specifically, one obtains Volterra integrodifferential equations, exemplified by

$$\text{(E)} \qquad \frac{dx}{dt} + \int_0^t a(t-s)g(x(s))ds = f(t) \ , \quad x(0) = x_0 \ , \quad t > 0 \ .$$

Here $f(t)$ is the external input, $a(t)$ is the feedback kernel, $g(x)$ is an in general nonlinear function of x and $x(t)$ is the state of the system at time t.

The key problem concerning (E) is the behavior of $x(t)$ for large values of t. In particular one is interested in whether $x(t)$ tends to zero when $t \to \infty$ or whether the system keeps oscillating. The present report analyzes these questions. We are in particular interested in the case when the feedback kernel is large, that is when $a(t)$ is not integrable over the positive half-axis. Examples of such kernels often occur in applications where one encounters kernels behaving roughly as $t^{-\alpha}$, for some $0 < \alpha < 1$, for large t.

The second key aspect of this report is that we do allow large input functions $f(t)$. We only assume $f(t) \to 0$, as $t \to \infty$ but do not take $|f|^p$ to be integrable over the positive half-axis for any $p < \infty$.

Our main result (Theorem 1) concerns the nonlinear version of (E). We give conditions under which bounded solutions of (E) decay to zero as $t \to \infty$. We also give a result on the linear version of (E).

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ASYMPTOTIC PROPERTIES OF VOLTERRA EQUATIONS WITH NONINTEGRABLE KERNELS

Stig-Olof Londen

INTRODUCTION

In this report we examine the asymptotic behavior of the solutions of the scalar Volterra equation

(1.1) $$x'(t) + (a * g(x))(t) = f(t) , \quad t > 0 , \quad x(0) = x_0 ,$$

where a,g,f are given functions, x is the solution and where $*$ denotes convolution. Our first result (Theorem 1) concerns the nonlinear case with both a and f big; thus $a \notin L^1(R^+)$ and f satisfying only (1.7) are not excluded. In Theorem 2 we examine the linear version of (1.1) under the same size conditions.

We begin by stating

<u>Theorem 1. Assume</u>

(1.2) $$g \in C(R) ,$$

(1.3) $$a \in BV(R^+) , \quad a(\infty) = 0 ,$$

(1.4) $$\text{Re } \hat{a} > 0 , \quad \omega \in R , \omega \neq 0$$

$$\underline{\text{with}} \quad \hat{a}(\omega) = 0 \quad \text{if} \quad \omega \in Z \overset{\text{def}}{=} \{\omega \mid \text{Re } \hat{a}(\omega) = 0 , \quad \omega \neq 0\},$$

(1.5) Z <u>is countable.</u>

<u>Suppose the differential resolvent</u> $r_a(t)$ <u>of</u> $a(t)$ <u>satifies</u>

(1.6)
$$\begin{aligned} &(i) && r_a \in L^1(R^+) \\ &(ii) && r_a' \in L^1(R^+) \\ &(iii) && t r_a'' \in L^1(R^+) . \end{aligned}$$

<u>and let</u> f <u>be such that</u>

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$$(1.7) \qquad f \in L^\infty(R^+), \quad \lim_{t\to\infty} f(t) = 0 .$$

Finally let

(1.8) $x \in (L^\infty \cap LAC)(R^+)$ *be a solution of* (1.1) *a.e. on* R^+.

Then

$$(1.9) \qquad \lim_{t\to\infty} g(x(t)) = 0 .$$

Above $\hat{a}(\omega) \overset{\text{def}}{=} \int_0^\infty e^{-i\omega t} a(t)dt$. By (1.3) $\hat{a}$ is well defined for $\omega \in R$, $\omega \neq 0$. The differential resolvent $r_a(t)$ is defined as the solution of

$$(1.10) \qquad r_a'(t) + (r_a * a)(t) = 0 , \quad r_a(0) = 1 .$$

The question naturally arises whether there exist kernels $a \notin L^1(R^+)$ and of positive type (i.e. satisfying (1.4)) such that (1.6) holds. By a well-known result [7] the conditions $i\omega + \hat{a}(\omega) \neq 0$, $\omega \in R$, and

(1.11) $a(t)$ positive, nonincreasing and convex on R^+,

imply (1.6i). But under the same conditions one does in fact have (1.6ii) and if moreover $-a'(t)$ is convex then (1.6iii) is true. The two last statements are contained in Lemma 1 below which is proved in Section 3.

Lemma 1. Let $a(t)$ *be nonnegative, nonincreasing and convex on* R^+ *with* $a(0) < \infty$, $a(\infty) = 0$, *and assume that for every fixed* $T > 0$ *one has that* $a(t)$ *is not linear in all the intervals* $[nT, (n+1)T]$; $n = 0,1,\cdots$. *Then* (1.4) *with* Z *empty and* (1.6i,ii) *hold. If in addition* $-a'(t)$ *is convex then* (1.6iii) *is true.*

Apart from being applicable to equations with kernels $a \notin L^1(R^+)$ the above theorem also extends recent work [5], [9] done for equations having kernels $a \in L^1(R^+)$ and a nonhomogeneous term f satisfying only (1.7). To see this we formulate Lemma 2 which follows upon an examination of the proof of Theorem 1, and the fact that $a \in L^1(R^+)$ together with (1.4) and $0 \notin Z$ implies $r_a, r'_a \in L^1(R^+)$.

<u>Lemma 2.</u> <u>Let</u> (1.2), (1.4), (1.5), (1.6iii), (1.7) <u>and</u> (1.8) <u>hold.</u> <u>In addition assume</u> $a \in L^1(R^+)$, <u>and</u> $0 \notin Z$. <u>Then</u> (1.9) <u>is true.</u>

Both [5] and [9] work with more general forms of the convolution term than (1.1). However, in addition to the condition (corresponding to) $a \in L^1(R^+)$ the results of [9] do require a moment condition on the kernel to be satisfied and the results of [5] require g to be locally Lipschitzian. At the price of (1.6iii) all this has been removed.

The proof of Theorem 1 is essentially based on (1.4), on the size conditions (1.6), and on an asymptotic result obtained in [9]. Observe that the countability assumption (1.5) can be dropped (without the addition of any other hypothesis) if one applies the technique of [4] to the integrated version of (1.1). As this has not been explicitly done we prefer to use [9, Theorem 1b].

One of the main problems concerning the linear equation

(1.12) $\quad x'(t) + (a * x)(t) = f(t) , \quad t \geq 0, \quad x(0) = x_0 ,$

is the formulation of hypotheses on the kernel $a(t)$ which imply $r_a \in L^1(R^+)$. The most frequently used approach to this hard problem is to give conditions on $a(t)$ which imply that $\hat{r}_a(\omega)$ is sufficiently smooth. But one also has (provided one at first demonstrates that in case f has compact support then $x \in L^\infty(R^+)$) that $r_a \in L^1(R^+)$ follows if the implication (1.13) is true [8] .

(1.13) $$\left\{ \begin{array}{l} x \in L^\infty(R^+) \text{ satisfies } (1.12) \\ \text{with } f \text{ satisfying } (1.7) \end{array} \right\} \text{ implies } \{\lim_{t\to\infty} x(t) = 0 \}$$

We show below that under reasonable conditions on the size of the derivatives of the kernel $a(t)$ one may weaken (1.13) to

$$\text{(1.14)} \qquad \left.\begin{array}{l} x \in L^{\infty}(R^{+}) \ \text{ satisfies (1.12)} \\ \text{with } f \ \text{ satisfying (1.7)} \\ \text{and } \lim\limits_{t\to\infty} x'(t) = 0 \end{array}\right\} \text{ implies } \{\lim_{t\to\infty} x(t) = 0\} ,$$

without altering the conclusion, namely that $r_a \in L^1(R^+)$ holds if the implication (1.14) is true.

A recent article by Gripenberg [2] analyzes the integral resolvent $R_a(t)$ of $a(t)$ by a related approach but under different hypotheses.

__Theorem 2.__ __Let__ $a(t) \in C^1[0,\infty)$, __with__ $a(\infty) = 0$ __satisfy__

$$\text{(1.15)} \qquad a' \in L^1(R^+)$$

$$\text{(1.16)} \qquad a' \in BV(R^+)$$

$$\text{(1.17)} \qquad \int_{R^+} t|da'(t)| < \infty$$

$$\text{(1.18)} \qquad \left\{\begin{array}{l} Z + \hat{a}(z) \neq 0 , \quad \operatorname{Re} Z > 0 , \quad Z \neq 0 \\ \\ \liminf\limits_{\substack{|Z|\to 0,\ Z\neq 0 \\ \operatorname{Re} Z > 0}} |Z + \hat{a}(Z)| > 0 , \end{array}\right.$$

__and suppose__ (1.14) __holds. Then__ $r_a \in L^1(R^+)$.

2. PROOF OF THEOREM 1 .

Convolve (1.1) with r_a and use the fact that r_a satisfies (1.10). This gives

$$(2.1) \qquad (r_a * x')(t) - (r_a' * g(x))(t) = (r_a * f)(t) \; .$$

An integration of the first term on the left side of (2.1) by parts results in

$$(2.2) \qquad x(t) - (r_a' * h(x))(t) = F(t)$$

where $h(x)$, $x \in R$; $F(t)$, $t \in R^+$; are defined by

$$(2.3) \qquad h(x) = g(x) - x \; , \qquad F(t) = (r_a * f)(t) + r_a(t)x_0 \; .$$

From (1.6i,ii), (1.7) and (1.10) follows

$$(2.4) \qquad F \in (L^\infty \cap LAC)(R^+), \qquad \lim_{t\to\infty} F(t) = 0$$

$$(2.5) \qquad F' \in L^\infty(R^+) \; , \qquad \lim_{t\to\infty} F'(t) = 0 \; .$$

Thus

$$(2.6) \qquad \lim_{t\to\infty} \int_0^t F'(\tau)d\tau = -F(0) = -x_0 \; .$$

Differentiate (2.2), use the fact that $r_a'(0) = 0$ and define $b(t) = -r_a''(t)$. This gives

$$(2.7) \qquad x'(t) + (b * h(x))(t) = F'(t) \; .$$

Then observe that $c(t) \overset{\text{def}}{=} \int_0^t b(s)ds = -r_a'(t)$ is such that

$$(2.8) \qquad c(\infty) = 0 \; , \qquad \int_{R^+} c(t)dt = 1 \; .$$

From the fact that $b \in L^1(R^+)$ and from (1.10) we have

$$(2.9) \qquad \hat{b}(\omega) = i\omega\hat{a}(\omega)[i\omega + \hat{a}(\omega)]^{-1} \; , \qquad \omega \neq 0 \; ; \qquad \hat{b}(0) = 0 \; .$$

Note that by (1.4) the transform condition $i\omega + \hat{a}(\omega) \neq 0$ is satisfied for $\omega \neq 0$. A simple computation gives

(2.10) $$\mathrm{Re}\,\hat{b}(\omega) = \omega^2 \mathrm{Re}\{\hat{a}(\omega)\}|i\omega + \hat{a}(\omega)|^{-2} \qquad \omega \neq 0$$

(2.11) $$\mathrm{Im}\,\hat{b}(\omega) = [\omega(\mathrm{Re}\,\hat{a})^2 + \omega^2 \mathrm{Im}\hat{a} + \omega(\mathrm{Im}\hat{a})^2]|i\omega + \hat{a}(\omega)|^{-2}, \quad \omega \neq 0,$$

and so, by (1.4), (2.9)-(2.11),

(2.12) $$\begin{cases} \mathrm{Re}\,\hat{b}(\omega) \geq 0, \quad \omega \in R; \quad \mathrm{Re}\,\hat{b}(\omega) = 0 \text{ iff } \omega \in Z \cup \{0\}, \\ \hat{b}(\omega) = 0 \quad \text{if} \quad \mathrm{Re}\,\hat{b}(\omega) = 0. \end{cases}$$

By (1.5), (1.6) - in particular we need (1.6iii) - (1.8), (2.4), (2.5), (2.12) and as $h \in C(R)$ we may apply [9, Theorem 1b] to get

(2.13) $$\lim_{t\to\infty} [x(t) + h(x(t)) \int_{R^+} c(\tau)d\tau] = 0.$$

But if the first part of (2.3) and the second part of (2.8) are used in (2.13) one gets (1.9).

3. PROOF OF LEMMA 1

It is well-known that under the stated hypotheses (1.4) holds with Z empty. See for example [10, p. 170], [3, p. 546]. From [7] we have that (1.6i) holds.

To show that (1.6ii) is satisfied one may at first observe that by (1.6i), (1.10), the monotonicity of $a(t)$ and as $a(0) < \infty$ one has $r_a' \in L^\infty(R^+)$. Thus we may Laplace transform r_a' for $\mathrm{Re}\, s > 0$ and obtain $\tilde{r}_a'(s) = -\tilde{a}(s)[s + \tilde{a}(s)]^{-1}$. (Note that $s + \tilde{a}(s) \neq 0$ for $\mathrm{Re}\, s > 0$). But by (1.3) $|\tilde{a}(s)| = O(|s|^{-1})$ as $|s| \to \infty$ and so we conclude by an application of [6, p. 368] that

$$r_a' \in L^2(R^+)\ ,\quad \hat{r}_a(\omega) = -\hat{a}(\omega)[i\omega + \hat{a}(\omega)]^{-1},\qquad \omega \neq 0\ . \tag{3.1}$$

To have $r_a' \in L^1(R^+)$ it suffices to show that $\hat{r}_a'(\omega)$ is locally absolutely continuous on R and satisfies

$$\frac{d}{d\omega}\hat{r}_a'(\omega) \in L^1(R)\ . \tag{3.2}$$

But (3.2) follows after straightforward computations which make use of the monotonicity of a and the estimates in [7, Lemma 1].

To prove (1.6iii) one notes at first that the Laplace transform of $c(t) \overset{\text{def}}{=} t\, r_a''(t)$ is the analytic function $\tilde{c}(s) = \Big[s^2 \dfrac{d\tilde{a}(s)}{ds} + [\tilde{a}(s)]^2\Big][s + \tilde{a}(s)]^{-2}$, $\mathrm{Re}\, s > 0$. We assert that

$$\sup_{0 < x < \infty} \int_R |\tilde{c}(x + iy)|^2 dy < \infty\ . \tag{3.3}$$

To see this one observes that

$$s^2 \frac{d\tilde{a}(s)}{ds} = -s\tilde{a}(s) - L\{ta''(t) - a'(t)\}\ , \tag{3.4}$$

that $|\tilde{a}(s)| = O(|s|^{-1})$ and that by the monotonicity of $a(t)$ one has ta'', $a' \in L^1(R^+)$. Consequently

$$\sup_{\mathrm{Re}\, s>0} \left|s^2 \frac{d\tilde{a}(s)}{ds}\right| < \infty \quad \text{and so} \quad \sup_{\mathrm{Re}\, s>0} |\tilde{c}(s)| < \infty ,$$

which together with the asymptotic estimate for $\tilde{a}(s)$ gives (3.3). Thus $c \in L^2(R^+)$ and

$$\hat{c}(\omega) = \frac{[\hat{a}(\omega)]^2 + i\omega^2 \frac{d\hat{a}}{d\omega}}{[\hat{a}(\omega) + i\omega]^2} , \quad \omega \neq 0 . \tag{3.5}$$

To have $c \in L^1(R^+)$ we need to verify that $\hat{c}$ is locally absolutely continuous and satisfies

$$\frac{d\hat{c}(\omega)}{d\omega} \in L^1(R) . \tag{3.6}$$

From [1, p. 972] we have (under the assumption that $-a'$ is convex) that $\hat{a}(\omega)$ is twice continuously differentiable and satisfies

$$\left|\frac{d^2\hat{a}(\omega)}{d\omega^2}\right| \leq K \int_0^{1/|\omega|} t^2\, a(t)dt , \quad \omega \neq 0 \tag{3.7}$$

for some constant K. Estimating this upwards gives

$$\left|\frac{d^2\hat{a}}{d\omega^2}\right| \leq K|\omega|^{-2} \int_0^{|\omega|^{-1}} a(t)dt \leq K_1|\omega|^{-2}\hat{a}(\omega) \tag{3.8}$$

where the second inequality follows from [7, Lemma 1]. Note that we also have

$$\left|\frac{d\hat{a}(\omega)}{d\omega}\right| \leq K|\omega|^{-1}|\hat{a}(\omega)| . \tag{3.9}$$

It now takes some computations which use (3.5), (3.8), (3.9) and the asymptotic estimates

$$\left|\frac{d^{(k)}\hat{a}(\omega)}{d\omega^k}\right| = O(|\omega|^{-k}) , \quad \omega \to \infty , \quad k = 0,1,2,$$

to arrive at (3.6).

4. PROOF OF THEOREM 2.

Let $\Gamma(x)$ denote the positive limit set of x, i.e.

$$\Gamma(x) = \{y \in L^\infty(R) \mid \text{ there exists } t_k \to \infty \text{ such that } x(t+t_k) \to y(t) \tag{4.1}$$
$$\text{weak}^* \text{ in } L^\infty(R)\}.$$

We show that

$$\text{every } y \in \Gamma(x) \text{ is a constant.} \tag{4.2}$$

Take $c > 0$ and define Λ_c by

$$\Lambda_c(\omega) = \begin{cases} 1 & |\omega| \leq c \\ 0 & |\omega| \geq 2c \\ 1 + c^{-1}[c - |\omega|], & c \leq |\omega| \leq 2c \end{cases} \tag{4.3}$$

There exists $\delta_c(t)$ such that

$$\delta_c \in L^1(R), \quad \hat{\delta}_c(\omega) = \Lambda_c(\omega). \tag{4.4}$$

Define, for $n = 1,2,\cdots$, and $t \geq 0$,

$$f_n(t) = e^{-\frac{t}{n}} f(t), \tag{4.5}$$

thus

$$f_n \in L^1(R^+). \tag{4.6}$$

Let x_n satisfy

$$x_n'(t) + (a * x_n)(t) = f_n(t), \quad x_n(0) = x_0, \tag{4.7}$$

then

$$x_n(t) = x_0 r_a(t) + (r_a * f_n)(t), \quad t \geq 0 \tag{4.8}$$

and

$$x_n \to x, \text{ as } n \to \infty, \text{ uniformly on compact sets of } R^+. \tag{4.9}$$

From (1.15), (1.16), (1.18) follows (much as in the proof of Lemma 1) that

$$r_a(t),\ \frac{dr_a}{dt} \in L^2(R^+)\ , \tag{4.10}$$

and so

$$\lim_{t\to\infty} r_a(t) = 0\ . \tag{4.11}$$

Define $x_n = x = f_n = f = 0$ for $t < 0$. By (4.6), (4.10) we may take Fourier transforms in (4.8) and obtain

$$\hat{x}_n(\omega) = x_0\hat{r}_a(\omega) + [i\omega + \hat{a}(\omega)]^{-1}\hat{f}_n(\omega)\ , \quad \omega \neq 0\ . \tag{4.12}$$

As $\hat{\delta}_c(\omega)[i\omega + \hat{a}(\omega)]^{-1} \in L^2(R)$ we have that there exists b_1 such that

$$b_1 \in L^2(R)\ , \quad \hat{b}_1(\omega) = \frac{\hat{\delta}_c(\omega)}{i\omega + \hat{a}(\omega)}\ . \tag{4.13}$$

Suppose for the moment that there exists $b_2(t)$ satisfying

$$b_2 \in L^1(R)\ , \quad \hat{b}_2(\omega) = [1 - \hat{\delta}_c(\omega)][i\omega + \hat{a}(\omega)]^{-1}\ . \tag{4.14}$$

Then

$$x_n(t) = x_0r_a(t) + \int_0^\infty b_1(t-s)f_n(s)ds + \int_0^\infty b_2(t-s)f_n(s)ds\ . \tag{4.15}$$

As $b_1 + b_2 = r = 0$ for $t < 0$ we have $b_1(t) = -b_2(t)$, $t < 0$. But

$$\left|\int_t^\infty b_2(t-s)f_n(s)ds\right| \le \|b_2\|_{L^1(R)}\|f\|_{L^\infty(R)}\ , \quad \forall\ t,n\ , \tag{4.16}$$

and so

$$\sup_{t,n}\left|\int_t^\infty b_1(t-s)f_n(s)ds\right| = K < \infty\ . \tag{4.17}$$

Also

$$\lim_{n\to\infty} \int_0^\infty b_2(t-s)f_n(s)ds = \int_0^\infty b_2(t-s)f(s)ds\ , \tag{4.18}$$

uniformly for $t \in R$. Define $g_n(t)$, $g(t)$, $t \in R$, by

(4.19) $g_n(t) = \int_0^\infty b_1(t-s)f_n(s)ds, \quad g(t) = x(t) - x_0 r_a(t) - \int_0^\infty b_2(t-s)f(s)ds.$

Then

$$g_n \in L^2(R) \ , \quad g \in L^\infty(R) \ , \tag{4.20}$$

and so g_n, g respresent tempered distributions.

We claim that

$$\lim_{n\to\infty} (g_n - g) = 0 \quad \text{weak}^* \ \text{in} \ S' . \tag{4.21}$$

From (4.9), (4.15), (4.18), (4.19)

$$\lim_{n\to\infty} (g_n(t) - g(t)) = 0 \quad \text{uniformly on compact sets of} \ R , \tag{4.22}$$

and as, by (4.17)

(4.23)

$$\begin{cases} |g_n(t)| = \left| \int_0^t + \int_t^\infty \{b_1(t-s)f_n(s)ds\} \right| \leq t^{1/2} \|b_1\|_{L^2(R)} \|f\|_{L^\infty(R)} + K \ , & t > 0 \ , \\ |g_n(t)| \leq K \ , & t < 0 \ , \end{cases}$$

we conclude that (4.21) holds. But then

$$\lim_{n\to\infty} (\hat{g}_n - \hat{g}) = 0 \quad \text{weak}^* \ \text{in} \ S' . \tag{4.24}$$

By (4.3), (4.4), (4.13), (4.19)

$$\text{supp } \hat{g}_n \subset [-2c, 2c]. \tag{4.25}$$

From (4.24), (4.25) follows

$$\text{supp } \hat{g} \subset [-2c, 2c] \ . \tag{4.26}$$

For $g \in L^\infty(R)$ we denote the spectrum of g (equivalently the support of the distribution Fourier transform) by $\sigma(g)$. The spectrum of a set A is defined as $\sigma(A) = \overline{\bigcup_{\varphi \in A} \sigma(\varphi)}$. For any $y \in L^\infty(R)$ we have $\sigma(\Gamma(y)) \subset \sigma(y)$ and therefore, by (4.26), (4.27) $$\sigma(\Gamma(g)) \subset \sigma(g) \subset [-2c, 2c].$$

But from (4.11) and the second part of (4.19) follows, as $b_2 \in L^1(R)$ and $f \to 0$ when $t \to \infty$,

$$\Gamma(g) = \Gamma(x)$$

and so $\sigma(\Gamma(g)) = \sigma(\Gamma(x))$. Hence $\sigma(\Gamma(x)) \subset [-2c, 2c]$. But c was arbitrary and therefore we conclude that $\sigma(\Gamma(x)) = 0$ which implies (4.2).

We return to the proof of (4.14). By (1.15), (1.18), (4.3), (4.4) we have that $[1 - \hat{\delta}_c(\omega)][i\omega + \hat{a}(\omega)]^{-1}$ is locally the transform of an L^1-function. Thus we only need to check the behavior of $\hat{b}_2$ at infinity. But if one rewrites $\hat{a}$ as

$$\hat{a}(\omega) = a(0)[i\omega]^{-1} - a'(0)\omega^{-2} - (d\hat{a}')\omega^{-2}, \quad \omega \neq 0$$

and uses (1.17) it follows after some calculations that

$$\frac{d}{d\omega}([i\omega + \hat{a}(\omega)]^{-1}) \in L^1((-\infty, -2c] \cup [2c, \infty))$$

and so (4.14) is true. The assertion (4.2) is hence valid.

From [8, Theorem 3.1] follows that the proof of Theorem 2 is complete provided we show that if f has compact support then

$$x_0 r_a(t) + (r_a * f)(t) \in L^\infty(R^+) .$$

This however is a consequence of (4.11).

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We study the asymptotic behavior of the solutions of the scalar Volterra integrodifferential equation

$$\text{(E)} \qquad x'(t) + (a * g(x))(t) = f(t), \quad t \geq 0, \quad x(0) = x_0 ,$$

where $a, f : R^+ \to R$ and $g : R \to R$ are given functions, $*$ denotes convolution and $x : R^+ \to R$ is the solution. We are in particular interested in

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ABSTRACT (continued)

the largely unsolved case when $a \notin L^1(R^+)$ and f vanishes at infinity but does not belong to any $L^p(R^+)$ space for $p < \infty$. The report examines both the linear $(g(x) \equiv x)$ and the nonlinear $(g(x) \neq x)$ version of (E).